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A remark on solutions of reflection equation for the critical $\mathbf{Z}_N$-symmetric vertex model

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Abstract

We consider solutions to the reflection equation for the critical $\mathbf{Z}_N$-symmetric vertex model, which is the trigonometric limit of the elliptic $\mathbf{Z}_N$-symmetric R-matrix of Belavin. These critical R-matrices have two parameters u and η. The transfer matrices $\mathcal{T}(u,\eta)$ constructed from this R-matrix $R(u,\eta)$ under the cyclic boundary condition are commutative among different u when η is in common, $[\mathcal{T}(u,\eta),\mathcal{T}(v,\eta)] = 0$. We prove that an arbitrary solution to the reflection equation is independent of the parameter η.

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1. Introduction

The Yang–Baxter equation

$$R^{01}(u)R^{02}(u+v)R^{12}(v) = R^{12}(v)R^{02}(u+v)R^{01}(u) \in \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N)$$

guarantees the commutativity of the transfer matrices $\mathcal{T}(u)$

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u)$$

under the cyclic boundary condition

$$R^{01}(u)R^{02}(u)\cdots R^{0l}(u) \in \mathrm{End}(\mathbf{C}^N \otimes \overbrace{\mathbf{C}^N \otimes \cdots \otimes \mathbf{C}^N}^{l\ \text{times}}).$$

$$\mathcal{T}(u) := \mathrm{tr}_0(R^{01}(u)R^{02}(u)\cdots R^{0l}(u)) \in \mathrm{End}(\overbrace{\mathbf{C}^N \otimes \cdots \otimes \mathbf{C}^N}^{l\ \text{times}}).$$

Sklyanin [1] proposed the reflection equation

$$\begin{aligned} &R^{12}(u_1-u_2)K^1(u_1)R^{21}(u_1+u_2)K^2(u_2) \\ &\quad = K^2(u_2)R^{12}(u_1+u_2)K^1(u_1)R^{21}(u_1-u_2) \in \mathrm{End}\,(\mathbf{C}^N \otimes \mathbf{C}^N) \end{aligned} \tag{1.1}$$

associated with a solution $R(u)$ to the Yang–Baxter equation. When the R-matrix is the one of Belavin's $\mathbf{Z}_N$-symmetric vertex model $R(u, \eta)$ [2], transfer matrices $\mathcal{T}_K(u)$ with a fixed boundary condition specified by $K(u)$, a solution to the reflection equation (1.1),

$$\mathcal{T}_K(u) := \mathrm{tr}_0(K_+(u)\mathcal{T}(-u)^{-1}K(u)\mathcal{T}(u))$$
$$K_+(u) = K(-u - N\eta/2)$$

also satisfies

$$\mathcal{T}_K(u)\mathcal{T}_K(v) = \mathcal{T}_K(v)\mathcal{T}_K(u)$$

so constitutes another commutative family of transfer matrices.

But we do not know much about solutions to the reflection equation up to now. When the original papers of Sklyanin and Kullish [1, 3] were published, they proposed $N = 2$ and diagonal K-matrix. Later Konno and Inami [4] and Hou and Yue [5] independently investigated solutions in the case of the eight-vertex model, and classified all elliptic solutions in this case. These works are all contained in the 'sl_2-theory'. Beyond sl_2, we have the results by Komori and Hikami [6], in which they proposed elliptic solutions with four parameters besides the spectral one for any Belavin $\mathbf{Z}_N$-symmetric vertex model, and the results of Ozaki [7] and Furutsu and Kojima [8] which list the trigonometric diagonal solutions. In the case of $N = 3$, neither of them give the complete trigonometric solution to the reflection equation, and is contained in the solution obtained in [13].

In this paper, we study solutions to the reflection equation in the case when the R-matrix is the trigonometric limit of the Belavin $\mathbf{Z}_N$-symmetric vertex model. We divide the matrix elements of the reflection equation into 15 groups and clarify the dependence among them, then we prove that any solution to the reflection equation (1.1) is independent of the parameter η of $R(u, \eta)$ for all N.

In section 2, we review the R-matrix of the $\mathbf{Z}_N$-symmetric vertex model of Belavin, and the necessary notation for trigonometric functions and matrices. We write up the components of the reflection equation, and divide them into 15 groups. Relations among these groups are shown, and here we prove the main result of this paper, theorems 2 and 3. We devote section 4 to discussions.

2. The Belavin $\mathbf{Z}_N$-symmetric vertex model

We fix the standard orthonormal basis $\{e_0, e_1, \ldots, e_{N-1}\}$ of the vector space $\mathbf{C}^N$, and extend the indices to all integers by defining $e_{j+N} = e_j$. The matrix element A^i_j of $A \in \mathrm{End}(\mathbf{C}^N)$ is defined by $Ae_j = \sum_{i=0}^{N-1} e_i A^i_j$, and two matrices g and h are also defined by

$$ge_j = \omega^j e_j \qquad he_j = e_{j+1}$$

where $\omega = \mathrm{e}^{2\sqrt{-1}\pi/N} = \mathbf{e}[2/N]$. They satisfy $gh = \omega hg$. We often abbreviate the exponential function $\mathrm{e}^{\sqrt{-1}\pi u}$ and the trigonometric functions $\sin \pi u$ and $\cos \pi u$ to $\mathbf{e}[u]$, $\mathbf{s}[u]$ and $\mathbf{c}[u]$, respectively. We define the theta function $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](u|\tau)$ by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(u|\,\tau) = \sum_{n\in\mathbf{Z}} \mathbf{e}[(n+a)^2\tau + 2(n+a)(u+b)]. \tag{2.1}$$

Let $q = \mathbf{e}[2\tau] = \mathrm{e}^{2\sqrt{-1}\pi\tau}$, then the theta function behaves in the limit of $\tau \to +\sqrt{-1}\infty$ as

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(u|\,\tau) = \begin{cases} q^{\varphi(a)^2}\mathbf{e}[2\varphi(a)(u+b)] + o\big(q^{\varphi(a)^2}\big) & \text{for} \quad a \not\equiv 1/2 \\ q^{1/4}\cdot 2\mathbf{c}[u+b] + o(q^{1/4}) & \text{for} \quad a \equiv 1/2 \end{cases}$$

where $\varphi(a) = a - [a + 1/2]$ and $[\alpha]$ denotes the largest integer not exceeding α.

Definition 1 ([2]). *The R-matrix of the critical $\mathbf{Z}_N$-symmetric vertex model of Belavin $R(u) \in \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N)$ is defined by*

$$R(u) = \lim_{\tau \to +\sqrt{-1}\infty} \frac{1}{N} \sum_{\alpha,\beta=0}^{N-1} w_{\alpha,\beta}(u, \eta|\tau) h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha}.$$

$$w_{\alpha,\beta}(u, \eta|\tau) = \vartheta \begin{bmatrix} \alpha/N + 1/2 \\ \beta/N + 1/2 \end{bmatrix} \left(u + \frac{\eta}{N} \middle| \tau\right) \cdot \left(\vartheta \begin{bmatrix} \alpha/N + 1/2 \\ \beta/N + 1/2 \end{bmatrix} \left(\frac{\eta}{N} \middle| \tau\right)\right)^{-1}.$$

The following symmetries of the Belavin R-matrix are immediate from the definition.

Proposition 1. *The critical $\mathbf{Z}_N$-symmetric vertex model of Belavin $R(u)$ has two symmetries,*

$$\mathbf{Z}_N\text{-symmetry} : R^{ij}_{kl}(u) = R^{i+p,j+p}_{k+p,l+p}(u) \qquad \text{for} \quad p \in \mathbf{Z}/N\mathbf{Z}$$

$$\text{Conservation law} : R^{ij}_{kl}(u) = \delta_{i+j,k+l} R^{ij}_{kl}(u)$$

which are equivalent to

$$(h \otimes h)^{-1} R(u)(h \otimes h) = (g \otimes g)^{-1} R(u)(g \otimes g) = R(u).$$

It satisfies the Yang–Baxter equation.

Theorem 1 ([9–11]). *The critical $\mathbf{Z}_N$-symmetric vertex model of Belavin $R(u)$ satisfies*

$$R^{01}(u) R^{02}(u+v) R^{12}(v) = R^{12}(v) R^{02}(u+v) R^{01}(u) \in \mathit{End}(\mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N)$$

where, for example, $R^{02}(u)$ acts as $R(u)$ on the zeroth and the second components of $\mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N$ and as an identity on the first component. (We are now counting from zero.)

According to these symmetries, the matrix elements $R^{ij}_{kl}(u)$ of $R(u)$ depend only on $i-k$ and $j-k$,

$$R^{ij}_{kl}(u) = \delta_{i+j,k+l} \cdot R^{i-k\,j-k}_{0\quad l-k}(u) =: \delta_{i+j,k+l} \cdot S^{i-k,j-k}(u). \tag{2.2}$$

For later use, we define

Definition 2.

$$a^* := a - N \left[\frac{a}{N}\right]$$

where a^* is the integer a in the interval $[0, N)$ which is congruent to a modulo N. The explicit form of $S^{ab}(u)$ is obtained as

Proposition 2 ([12]).

$$S^{ab}(u) = \begin{cases} \dfrac{\mathbf{s}[u+\eta]}{\mathbf{s}[\eta]} & \textit{for} \quad a \equiv b \equiv 0 \\ \mathbf{e}\left[2\left(\dfrac{1}{2} - \dfrac{a^*}{N}\right) u\right] & \textit{for} \quad a \equiv 1, 2, \ldots, N-1 \quad b \equiv 0 \\ \mathbf{e}\left[2\left(\dfrac{b^*}{N} - \dfrac{1}{2}\right) \eta\right] \dfrac{\mathbf{s}[u]}{\mathbf{s}[\eta]} & \textit{for} \quad a \equiv 0, \quad b \equiv 1, 2, \ldots, N-1 \\ 0 & \textit{for} \quad a, b \equiv 1, 2, \ldots, N-1 \end{cases}. \tag{2.3}$$

3. Reflection equation

Definition 3 (reflection equation). *The reflection equation is*

$$R^{12}(u_1-u_2)K^1(u_1)R^{21}(u_1+u_2)K^2(u_2)$$
$$= K^2(u_2)R^{12}(u_1+u_2)K^1(u_1)R^{21}(u_1-u_2) \in End(\mathbf{C}^N\otimes\mathbf{C}^N) \tag{3.1}$$

where

$$R^{12}(u)=R(u) \qquad R^{21}(u)=PR(u)P \qquad P(x\otimes y)=y\otimes x \quad \text{for any } x,y\in\mathbf{C}^N$$
$$K^1(u)=K(u)\otimes Id \qquad K^2(u)=Id\otimes K(u).$$

We will consider solutions $K(u)\in \mathrm{End}(\mathbf{C}^N)$ to the reflection equation (3.1) associated with the critical $\mathbf{Z}_N$-symmetric vertex model of Belavin $R(u)$ defined in (2.2) and (2.3). We define the elements of $K(u)\in \mathrm{End}(\mathbf{C}^N)$ by

$$K(u)^i_j = c^i_j(u)$$

and substituting (2.2) into the reflection equation (3.1), we have

Proposition 3. *The* $(m_1m_2|i_1i_2)$ *component of the reflection equation is*

$$\sum_{j,l\in\mathbf{Z}/N\mathbf{Z}} S^{l-m_2,l-m_1}(u_1-u_2)S^{l-j,i_1-l}(u_1+u_2)c^{m_1+m_2-l}_{i_1+j-l}(u_1)c^j_{i_2}(u_2)$$
$$=\sum_{j,l\in\mathbf{Z}/N\mathbf{Z}} S^{j-i_2,i_1-j}(u_1-u_2)S^{j_2-l,j-m_1}(u_1+u_2)c^{m_1+l-j}_{i_1+i_2-j}(u_1)c^{m_2}_l(u_2) \tag{3.2}$$

where $i_1,i_2,m_1,m_2\in\mathbf{Z}/N\mathbf{Z}$.

We will abbreviate $S^{ab}(u_1-u_2)$ and $S^{ab}(u_1+u_2)$ simply to $S^{ab}(-)$ and $S^{ab}(+)$, respectively,

$$S^{ab}(-):=S^{ab}(u_1-u_2) \qquad S^{ab}(+):=S^{ab}(u_1+u_2).$$

And we will also employ the shorthand notation c^{ac}_{bd} for the product of $c^a_b(u_1)$ and $c^c_d(u_2)$ in this order,

$$c^{ac}_{bd}:=c^a_b(u_1)c^c_d(u_2).$$

For example, (3.2) is rewritten in this notation as

$$\sum_{jl} S^{l-m_2,l-m_1}(-)S^{l-j,i_1-l}(+)c^{m_1+m_2-l,j}_{i_1+j-l\quad,i_2} = \sum_{jl} S^{j-i_2,i_1-j}(-)S^{j_2-l,j-m_1}(+)c^{m_1+l-j,m_2}_{i_1+i_2-j,l}. \tag{3.3}$$

Substituting (2.3)

$$S^{ab}(u)=\delta_{a0}S^{0b}(u)+\delta_{b0}S^{a0}(u)-\delta_{a0}\delta_{b0}S^{00}(u)$$

into the $(m_1 m_2|i_1 i_2)$ component of the reflection equation (3.3), after some calculation we have

$$
\begin{aligned}
&S^{0,m_2-m_1}(-)S^{0,i_1-m_2}(+)c_{i_1\ i_2}^{m_1 m_2} + S^{m_1-m_2,0}(-)S^{0,i_1-m_1}(+)c_{i_1\ i_2}^{m_2 m_1}\\
&\quad - S^{0,i_1-i_2}(-)S^{0,i_2-m_1}(+)c_{i_1\ i_2}^{m_1 m_2} - S^{i_1-i_2,0}(-)S^{0,i_1-m_1}(+)c_{i_2\ i_1}^{m_1 m_2}\\
&\quad + \delta_{m_1,i_1}\left\{\begin{aligned}&\sum_{j=0}^{N-1} S^{i_1-m_2,0}(-)S^{i_1-j,0}(+)c_{j\ \ i_2}^{m_2 j} - S^{i_1-m_2,0}(-)S^{0,0}(+)c_{i_1\ i_2}^{m_2 i_1}\\ &-\sum_{j=0}^{N-1} S^{i_1-i_2,0}(-)S^{i_1-j,0}(+)c_{i_2 j}^{j\ m_2} + S^{i_1-i_2,0}(-)S^{0,0}(+)c_{i_2 i_1}^{i_1 m_2}\end{aligned}\right\}\\
&\quad + \delta_{m_2,i_1}\left\{\sum_{j=0}^{N-1} S^{0,i_1-m_1}(-)S^{i_1-j,0}(+)c_{j\ \ i_2}^{m_1 j} - S^{0,i_1-m_1}(-)S^{0,0}(+)c_{i_1\ i_2}^{m_1 i_1}\right\}\\
&\quad + \delta_{m_1,i_2}\left\{-\sum_{j=0}^{N-1} S^{0,i_1-i_2}(-)S^{i_2-j,0}(+)c_{i_1 j}^{j\ m_2} + S^{0,i_1-i_2}(-)S^{0,0}(+)c_{i_1 i_2}^{i_2 m_2}\right\}\\
&\quad + \delta_{m_1,m_2}\left\{-S^{0,0}(-)S^{0,i_1-m_1}(+)c_{i_1\ i_2}^{m_1 m_1}\right\}\\
&\quad + \delta_{i_1,i_2}\left\{S^{0,0}(-)S^{0,i_1-m_1}(+)c_{i_1\ i_2}^{m_1 m_2}\right\}\\
&\quad + \delta_{m_1,m_2}\cdot\delta_{m_2,i_1}\left\{-\sum_{j=0}^{N-1} S^{0,0}(-)S^{i_1-j,0}(+)c_{j\ i_2}^{i_1 j} + S^{0,0}(-)S^{0,0}(+)c_{i_1 i_2}^{i_1 i_1}\right\}\\
&\quad + \delta_{m_1,i_1}\cdot\delta_{i_1,i_2}\left\{\sum_{j=0}^{N-1} S^{0,0}(-)S^{i_1-j,0}(+)c_{i_1 j}^{j\ m_2} - S^{0,0}(-)S^{0,0}(+)c_{i_1 i_1}^{i_1 m_2}\right\} = 0.
\end{aligned}
\tag{3.4}
$$

The reflection equation is a matrix equation in $\mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N)$. Its components are represented by four indices $i_1, i_2, m_1, m_2 \in \mathbf{Z}/N\mathbf{Z}$ as above, so it consists of N^4 scalar equations. We divide them into 15 groups according to the situation of the coincidence of these four indices $i_1, i_2, m_1, m_2 \in \mathbf{Z}/N\mathbf{Z}$. We define

$$
\text{Equation (I)} := \left\{\begin{array}{l|l}(m_1 m_2|i_1 i_2)\text{-component} & \text{where } m_1, m_2, i_1, i_2 \text{ are}\\ \text{of the reflection equation} & \text{all different from each other}\end{array}\right\}
$$

$$
\text{Equation (II)} := \left\{\begin{array}{l|l}(m_1 m_2|i_1 i_2)\text{-component} & \text{where } i_1 = i_2 \text{ and } m_1, m_2, i_1 \text{ are}\\ \text{of the reflection equation} & \text{different from each other}\end{array}\right\}
$$

$$
\text{Equation (III)} := \left\{\begin{array}{l|l}(m_1 m_2|i_1 i_2)\text{-component} & \text{where } m_1 = m_2 \text{ and } m_2, i_1, i_2 \text{ are}\\ \text{of the reflection equation} & \text{different from each other}\end{array}\right\}.
$$

Equation (I) consists of $N(N-1)(N-2)(N-3)$ scalar equations, and both equations (II) and (III) consist of $N(N-1)(N-2)$ scalar equations. We write down the definitions for the rest in a simpler way.

$$\text{Equation (IV)} = \left\{\begin{matrix} m_1 = i_1 \text{ and } m_1, m_2, i_2 \text{ are} \\ \text{different from each other}\end{matrix}\right\} \qquad \text{Equation (X)} := \{m_1 = i_2 \neq m_2 = i_1\}$$

$$\text{Equation (V)} := \left\{\begin{matrix} m_2 = i_2 \text{ and } m_1, m_2, i_1 \text{ are} \\ \text{different from each other}\end{matrix}\right\} \qquad \text{Equation (XI)} := \{m_1 = i_1 = i_2 \neq m_2\}$$

$$\text{Equation (VI)} := \left\{\begin{matrix} m_2 = i_1 \text{ and } m_1, m_2, i_2 \text{ are} \\ \text{different from each other}\end{matrix}\right\} \qquad \text{Equation (XII)} := \{m_1 = m_2 = i_1 \neq i_2\}$$

$$\text{Equation (VII)} := \left\{\begin{matrix} m_1 = i_2 \text{ and } m_1, m_2, i_1 \text{ are} \\ \text{different from each other}\end{matrix}\right\} \qquad \text{Equation (XIII)} := \{m_1 = m_2 = i_2 \neq i_1\}$$

$$\text{Equation (VIII)} := \{m_1 = m_2 \neq i_1 = i_2\} \qquad \text{Equation (XIV)} := \{m_1 \neq m_2 = i_1 = i_2\}$$

$$\text{Equation (IX)} := \{m_1 = i_1 \neq m_2 = i_2\} \qquad \text{Equation (XV)} := \{m_1 = m_2 = i_1 = i_2\}\,.$$

If we write the equations in equation (VIII), namely the case of $m_1 = m_2 \neq i_1 = i_2$ in (3.4), we find that they all vanish.

Lemma 1 (Equation (VIII))**.** *Equation (VIII) consists of trivial equations.*

To deal with other groups, we have to substitute the explicit form of $S^{ab}(u)$ in (2.3) into the components of the reflection equation. We prepare some notation here to express the results.

Definition 4. *For any three integers a, b and c, we define $P(a, b, c)$ by*

$$P(a, b, c) := \frac{1}{N}((a-b)^* + (b-c)^* + (c-a)^* - N).$$

We omit the proof of the following lemma.

Lemma 2. *For any integer a, we have*

$$a + (-a)^* = \begin{cases} N & \textit{for} \quad a \not\equiv 0 \mod N \\ 0 & \textit{for} \quad a \equiv 0 \mod N. \end{cases}$$

Three mutually different integers a_1, a_2 and a_3 uniquely define an element σ in the symmetric group $\mathcal{S}_3$ such that $a_{\sigma(1)} < a_{\sigma(2)} < a_{\sigma(3)}$. We have

$$P(a_1, a_2, a_3) = \begin{cases} 1 & \textit{for} \quad \mathrm{sgn}(\sigma) = 1 \\ 0 & \textit{for} \quad \mathrm{sgn}(\sigma) = -1 \end{cases}$$

consequently

$$\begin{aligned} P(a_1, a_2, a_3) &= P(a_3, a_1, a_2) = P(a_2, a_3, a_1) \\ &\neq P(a_3, a_2, a_1) = P(a_1, a_3, a_2) = P(a_2, a_1, a_3). \end{aligned}$$

Definition 5.

$$\tilde{A}(\alpha, \beta, \gamma | u_1, u_2) = \sum_{j=0}^{N-1} S^{\alpha-j,0}(+) c^{\beta\, j}_{j\, \gamma}$$

$$\tilde{B}(\alpha, \beta, \gamma | u_1, u_2) = \sum_{j=0}^{N-1} S^{\alpha-j,0}(+) c^{j\, \gamma}_{\beta\, j}$$

$$A(\alpha, \beta, \gamma | u_1, u_2) = \sum_{j=0}^{N-1} \mathbf{e}\left[-\frac{2}{N}(\alpha - j)^*(u_1 + u_2)\right] c^{\beta\, j}_{j\, \gamma}$$

$$B(\alpha,\beta,\gamma|u_1,u_2)=\sum_{j=0}^{N-1}\mathbf{e}\left[-\frac{2}{N}(\alpha-j)^*(u_1+u_2)\right]c_{\beta\,j}^{j\,\gamma}.$$

As we are abbreviating $c_j^i(u_1)c_l^k(u_2)$ to $c_{j\,l}^{i\,k}$ while keeping the order of u_1 and u_2, we also write $\tilde{A}(\alpha,\beta,\gamma|u_1,u_2)$ and $A(\alpha,\beta,\gamma|u_1,u_2)$ as

$$\tilde{A}(\alpha,\beta,\gamma):=\tilde{A}(\alpha,\beta,\gamma|u_1,u_2)\qquad A(\alpha,\beta,\gamma):=A(\alpha,\beta,\gamma|u_1,u_2)$$

for short.

Lemma 3.

$$\tilde{A}(\alpha,\beta,\gamma)=\mathbf{e}\,[u_1+u_2]\,A(\alpha,\beta,\gamma)+\frac{\mathbf{e}\,[-\eta]}{\mathbf{s}\,[\eta]}\mathbf{s}\,[u_1+u_2]\,c_{\alpha\,\gamma}^{\beta\,\alpha}$$

$$\tilde{B}(\alpha,\beta,\gamma)=\mathbf{e}\,[u_1+u_2]\,B(\alpha,\beta,\gamma)+\frac{\mathbf{e}\,[-\eta]}{\mathbf{s}\,[\eta]}\mathbf{s}\,[u_1+u_2]\,c_{\beta\,\alpha}^{\alpha\,\gamma}.$$

Here we list the explicit forms of the components of the reflection equations. In the notation defined so far and

$$u_{12}:=u_1-u_2\qquad v_{12}:=u_1+u_2 \tag{3.5}$$

they are

Equation (I) (m_1,m_2,i_1,i_2) : $\quad (P(m_1,i_1,m_2)-P(m_1,i_1,i_2))\,(1-\mathbf{e}\,[-2u_{12}])\,c_{i_1\;i_2}^{m_1m_2}$

$$+\mathbf{e}\left[-\frac{2}{N}(m_1-m_2)^*u_{12}\right]c_{i_1\;i_2}^{m_2m_1}-\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]c_{i_2\;i_1}^{m_2m_1}=0$$

Equation (II) (m_1,m_2,i) : $\quad (P(m_2,m_1,i)\mathbf{e}\,[-2u_{12}]+(1-P(m_2,m_1,i)))c_{i\;\;i}^{m_1m_2}$

$$=\mathbf{e}\left[-\frac{2}{N}(m_1-m_2)^*u_{12}\right]c_{i\;\;i}^{m_2m_1}$$

Equation (III) (m,i_1,i_2) : $\quad (P(i_1,i_2,m)\mathbf{e}\,[-2u_{12}]+(1-P(i_1,i_2,m)))c_{i_1i_2}^{m\,m}$

$$=\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]c_{i_2i_1}^{m\,m}$$

Equation (IV) (m,i_1,i_2) : $\quad \mathbf{e}\left[-\frac{2}{N}(i_1-m)^*u_{12}\right]A(i_1,m,i_2)$

$$-\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]B(i_1,i_2,m)+\frac{\mathbf{e}\,[-\eta]}{\mathbf{s}\,[\eta]}\mathbf{e}\,[-v_{12}]\,\mathbf{s}\,[v_{12}]$$

$$\times\left(\mathbf{e}\left[-\frac{2}{N}(i_1-m)^*u_{12}\right]c_{i_1i_2}^{m\,i_1}-\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]c_{i_2i_1}^{i_1m}\right)=0$$

Equation (V) (m,i_1,i_2) : $\quad \mathbf{e}\left[-\frac{2}{N}(m-i_2)^*u_{12}\right]c_{i_1i_2}^{i_2m}=\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]c_{i_2i_1}^{m\,i_2}$

Equation (VI) (m,i_1,i_2) : $\quad \mathbf{s}\,[v_{12}]\,\mathbf{e}\,[u_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(m-i_1)^*u_{12}\right]c_{i_1i_2}^{i_1m}\right.$

$$\left.-\mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*u_{12}\right]c_{i_2i_1}^{m\,i_1}\right)-2\sqrt{-1}P(i_1,i_2,m)$$

$$\times\,\mathbf{s}\,[u_{12}]\,\mathbf{s}\,[v_{12}]\,c_{i_1\,i_2}^{m\,i_1}+\mathbf{s}\,[u_{12}]\,\mathbf{e}\,[v_{12}]\,A(i_1,m,i_2)=0$$

Equation (VII) (m, i_1, i_2) : $\mathbf{s}[v_{12}]\,\mathbf{e}[u_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(i_1 - i_2)^* u_{12}\right] c^{i_2 m}_{i_2 i_1}\right.$

$$\left. - \mathbf{e}\left[-\frac{2}{N}(i_2 - m_2)^* u_{12}\right] c^{m i_2}_{i_1 i_2}\right) - 2\sqrt{-1}P(i_2, i_1, m)$$

$$\times\, \mathbf{s}[u_{12}]\,\mathbf{s}[v_{12}]\, c^{i_2 m}_{i_1 i_2} + \mathbf{s}[u_{12}]\,\mathbf{e}[v_{12}]\, B(i_2, i_1, m) = 0$$

Equation (VIII) (m, i) : $0 = 0$ trivial!

Equation (IX) (i_1, i_2) : $A(i_1, i_2, i_2) - B(i_1, i_2, i_2)$

$$+ \frac{\mathbf{e}[-\eta]}{\mathbf{s}[\eta]}\mathbf{e}[-v_{12}]\,\mathbf{s}[v_{12}]\left(c^{i_2 i_1}_{i_1 i_2} - c^{i_1 i_2}_{i_2 i_1}\right) = 0$$

Equation (X) (i_1, i_2) : $\mathbf{e}[u_{12}]\,\mathbf{s}[v_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(i_2 - i_1)^* u_{12}\right] c^{i_1 i_2}_{i_1 i_2}\right.$

$$\left. - \mathbf{e}\left[-\frac{2}{N}(i_1 - i_2)^* u_{12}\right] c^{i_2 i_1}_{i_2 i_1}\right) + \mathbf{s}[u_{12}]\,\mathbf{e}[v_{12}]\,(A(i_1, i_2, i_2) - B(i_2, i_1, i_1)) = 0$$

Equation (XI) (m, i) : $\mathbf{e}[v_{12}]\left(\mathbf{s}[\eta]\,\mathbf{e}\left[\frac{2}{N}(m - i)^* u_{12}\right] A(i, m, i)\right.$

$$\left. - \mathbf{e}[u_{12}]\,\mathbf{s}[u_{12} + \eta]\, B(i, i, m)\right) - \mathbf{s}[v_{12}]\,\mathbf{e}[-\eta]$$

$$\times \left(c^{i m}_{i i} - \mathbf{e}\left[\frac{2}{N}(m - i)^* u_{12}\right] c^{m i}_{i\ i}\right) = 0$$

Equation (XII) (m, i) : $\mathbf{e}[v_{12}]\left(\mathbf{s}[\eta]\,\mathbf{e}\left[\frac{2}{N}(i - m)^* u_{12}\right] B(m, i, m)\right.$

$$\left. - \mathbf{e}[u_{12}]\,\mathbf{s}[u_{12} + \eta]\, A(m, m, i)\right)$$

$$- \mathbf{s}[v_{12}]\,\mathbf{e}[-\eta]\left(c^{m m}_{m i} - \mathbf{e}\left[\frac{2}{N}(i - m)^* u_{12}\right] c^{m m}_{i\ m}\right) = 0$$

Equation (XIII) (m, i) : $\mathbf{s}[v_{12}]\left(c^{m m}_{i\ m} - \mathbf{e}\left[-\frac{2}{N}(i - m)^* u_{12}\right] c^{m m}_{m i}\right)$

$$- \mathbf{s}[u_{12}]\,\mathbf{e}[2u_2]\, B(m, i, m) = 0$$

Equation (XIV) (m, i) : $\mathbf{s}[v_{12}]\left(c^{m i}_{i\ i} - \mathbf{e}\left[-\frac{2}{N}(m - i)^* u_{12}\right] c^{i m}_{i i}\right)$

$$- \mathbf{s}[u_{12}]\,\mathbf{e}[2u_2]\, A(i, m, i) = 0$$

Equation (XV) (i) : $A(i, i, i) - B(i, i, i) = 0.$

Lemma 4 (equation (XI))**.** *If $K(u) = \left(c^a_b(u)\right)_{ab}$ satisfies equation (XIV), then it also satisfies equation (XI).*

Proof of lemma 4. Any element of equation (XIV) has the form

$$\mathbf{s}[u_{12}]\,\mathbf{e}[v_{12}]\, A(i, m, i) = \mathbf{e}[u_{12}]\,\mathbf{s}[v_{12}]\left(c^{m i}_{i\ i} - \mathbf{e}\left[-\frac{2}{N}(m - i)^* u_{12}\right] c^{i m}_{i i}\right) \tag{3.6}$$

for $m \neq i$. When we interchange u_1 and u_2, then u_{12}, v_{12}, c^{ac}_{bd} and $A(i, j, k)$ are altered into

$$\begin{aligned} u_{12|u_1 \leftrightarrow u_2} &= -u_{12} \qquad & v_{12|u_1 \leftrightarrow u_2} &= v_{12} \\ c^{ac}_{bd|u_1 \leftrightarrow u_2} &= c^{ca}_{db} \qquad & A(i, j, k)_{|u_1 \leftrightarrow u_2} &= B(i, k, j) \end{aligned} \tag{3.7}$$

and (3.6) becomes

$$\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[v_{12}\right]B(i,i,m) = -\mathbf{e}\left[-u_{12}\right]\mathbf{s}\left[v_{12}\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right). \tag{3.8}$$

After substituting $A(i,m,i)$ in (3.6) and $B(i,i,m)$ in (3.8) into equation (XI)

$$\mathbf{s}\left[v_{12}\right]\mathbf{e}\left[-\eta\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right) - \mathbf{e}\left[v_{12}\right]$$
$$\times\left(\mathbf{s}\left[\eta\right]\mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]A(i,m,i) - \mathbf{e}\left[u_{12}\right]\mathbf{s}\left[u_{12}+\eta\right]B(i,i,m)\right) = 0$$

then we have

$$\begin{aligned}
&\mathbf{s}\left[u_{12}\right]\cdot\mathbf{s}\left[v_{12}\right]\mathbf{e}\left[-\eta\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right)\\
&\quad -\mathbf{s}\left[u_{12}\right]\cdot\mathbf{e}\left[v_{12}\right]\mathbf{s}\left[\eta\right]\mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]A(i,m,i)\\
&\quad +\mathbf{s}\left[u_{12}\right]\cdot\mathbf{e}\left[v_{12}\right]\mathbf{e}\left[u_{12}\right]\mathbf{s}\left[u_{12}+\eta\right]B(i,i,m)\\
&= \mathbf{s}\left[u_{12}\right]\mathbf{s}\left[v_{12}\right]\mathbf{e}\left[-\eta\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right)\\
&\quad -\mathbf{s}\left[\eta\right]\mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]\cdot\left[\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[v_{12}\right]A(i,m,i)\right]\\
&\quad +\mathbf{e}\left[u_{12}\right]\mathbf{s}\left[u_{12}+\eta\right]\cdot\left[\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[v_{12}\right]B(i,i,m)\right]\\
&= \mathbf{s}\left[u_{12}\right]\mathbf{s}\left[v_{12}\right]\mathbf{e}\left[-\eta\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right)\\
&\quad -\mathbf{s}\left[\eta\right]\mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]\cdot\left[\mathbf{e}\left[u_{12}\right]\mathbf{s}\left[v_{12}\right]\left(c_{i\ i}^{m\,i} - \mathbf{e}\left[-\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\,i}^{i\,m}\right)\right]\\
&\quad -\mathbf{e}\left[u_{12}\right]\mathbf{s}\left[u_{12}+\eta\right]\cdot\left[\mathbf{e}\left[-u_{12}\right]\mathbf{s}\left[v_{12}\right]\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right)\right]\\
&= \mathbf{s}\left[v_{12}\right]\left(\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[-\eta\right]+\mathbf{s}\left[\eta\right]\mathbf{e}\left[u_{12}\right]-\mathbf{s}\left[u_{12}+\eta\right]\right)c_{i\,i}^{i\,m}\\
&\quad -\mathbf{s}\left[v_{12}\right]\mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]\left(\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[-\eta\right]+\mathbf{s}\left[\eta\right]\mathbf{e}\left[u_{12}\right]-\mathbf{s}\left[u_{12}+\eta\right]\right)c_{i\ i}^{m\,i}\\
&= \mathbf{s}\left[v_{12}\right]\left(\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[-\eta\right]+\mathbf{s}\left[\eta\right]\mathbf{e}\left[u_{12}\right]-\mathbf{s}\left[u_{12}+\eta\right]\right)\\
&\quad \times\left(c_{i\,i}^{i\,m} - \mathbf{e}\left[\frac{2}{N}(m-i)^{*}u_{12}\right]c_{i\ i}^{m\,i}\right).
\end{aligned}$$

Because the second factor of the last line is zero,

$$\begin{aligned}
&\mathbf{s}\left[u_{12}\right]\mathbf{e}\left[-\eta\right]+\mathbf{s}\left[\eta\right]\mathbf{e}\left[u_{12}\right]-\mathbf{s}\left[u_{12}+\eta\right]\\
&\quad = \frac{1}{2\sqrt{-1}}\begin{pmatrix}(\mathbf{e}\left[u_{12}\right]-\mathbf{e}\left[-u_{12}\right])\mathbf{e}\left[-\eta\right]+(\mathbf{e}\left[\eta\right]-\mathbf{e}\left[-\eta\right])\mathbf{e}\left[u_{12}\right]\\ -(\mathbf{e}\left[u_{12}+\eta\right]-\mathbf{e}\left[-u_{12}-\eta\right])\end{pmatrix} = 0
\end{aligned}$$

equation (XI) follows from equation (XIV). □

Almost the same arguments also show

Lemma 5 (equation (XII))**.** *If $K(u) = \left(c_b^a(u)\right)_{ab}$ satisfies equation (XIII), then it also satisfies equation (XII).*

Definition 6. *If two functions $f(u)$ and $g(u)$ satisfy*

$$f(u_1)g(u_2) = f(u_2)g(u_1)$$

for arbitrary u_1 and u_2, then we write

$$f(u) \sim g(u).$$

This relation '$\sim$' is reflexive and symmetric but not transitive, so it is not an equivalence relation. Any function $f(u)$ satisfies $f(u) \sim \mathbf{0}(u)$, where $\mathbf{0}(u)$ denotes the identically zero function.

Lemma 6 (equations (IX) and (XV))**.** *That $K(u) = \left(c_b^a(u)\right)_{ab}$ satisfies equation (IX) and equation (XV) is equivalent to that*

$$c_b^a(u) \sim c_a^b(u)$$

for any $a, b \in \mathbf{Z}/N\mathbf{Z}$.

Proof of lemma 6. When we rewrite equation (IX) in terms of

$$\gamma_{j,b} := c_{jb}^{bj} - c_{bj}^{jb}$$

we have

$$\sum_{j=0}^{N-1} \mathbf{e}\left[-\frac{2}{N}(i_1 - j)^* v_{12}\right] \gamma_{j,i_2} = -\frac{\mathbf{e}[-\eta]}{\mathbf{s}[\eta]} \mathbf{e}[-v_{12}]\, \mathbf{s}[v_{12}]\, \gamma_{i_1,i_2} \qquad (i_1, i_2 \in \mathbf{Z}/N\mathbf{Z}, i_1 \neq i_2).$$

The equations in equation (XV) are expressed in the same formula by setting $i_1 = i_2$

$$\sum_{j=0}^{N-1} \mathbf{e}\left[-\frac{2}{N}(i_2 - j)^* v_{12}\right] \gamma_{j,i_2} = 0.$$

If we fix $i_2 = i$ and consider the N equations for $i_1 = 0, 1, 2, \ldots, N-1$, then we have

$$\begin{pmatrix} 1-f & x^{N-1} & x^{N-2} & \cdots & x \\ x & 1-f & x^{N-1} & \cdots & x^2 \\ x^2 & x & 1-f & \cdots & x^3 \\ \vdots & \vdots & & \ddots & \vdots \\ x^{N-1} & x^{N-2} & \cdots & x & 1-f \end{pmatrix} \begin{pmatrix} \gamma_{0i} \\ \gamma_{1i} \\ \gamma_{2i} \\ \vdots \\ \gamma_{N-1i} \end{pmatrix} = 0 \qquad (i \in \mathbf{Z}/N\mathbf{Z})$$

where

$$x = \mathbf{e}\left[-\frac{2}{N} v_{12}\right] \qquad f = \frac{\mathbf{e}[-\eta]}{\mathbf{s}[\eta]} \mathbf{e}[-v_{12}]\, \mathbf{s}[v_{12}].$$

The determinant Δ of the above coefficient matrix becomes

$$\Delta = (1 - x^N)^N \prod_{k=0}^{N-1} \left(\frac{1}{\mathbf{e}[2\eta] - 1} + \frac{1}{1 - \omega^k x}\right) \qquad (\omega = \mathbf{e}[2/N])$$

and obviously not generically zero. We can conclude that if $K(u) = \left(c_b^a(u)\right)_{ab}$ satisfies equations (IX) and (XV), then we have

$$\gamma_{ab} = c_a^b(u_1)c_b^a(u_2) - c_a^b(u_2)c_b^a(u_1) \equiv 0 \qquad (a, b \in \mathbf{Z}/N\mathbf{Z}).$$

The converse is immediate. □

Equation (X) consists of $N(N-1)$ equations indexed by $i_1, i_2 \in \{0, 1, 2, \ldots, N-1\}$ $(i_1 \neq i_2)$,

$$\mathbf{e}[u_{12}]\,\mathbf{s}[v_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(i_2-i_1)^* u_{12}\right] c_{i_1 i_2}^{i_1 i_2} - \mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^* u_{12}\right] c_{i_2 i_1}^{i_2 i_1}\right)$$
$$+\,\mathbf{s}[u_{12}]\,\mathbf{e}[v_{12}]\,(A(i_1, i_2, i_2) - B(i_2, i_1, i_1)) = 0. \tag{3.9}$$

When we interchange u_1 and u_2 in (3.9) and make use of (3.7), we have the equation which is obtained after we interchange i_1 and i_2 in (3.9).

Definition 7.

$$\textit{Equation (X)}_{1/2} := \left\{ \begin{array}{l|l} (m_1 m_2 | i_1 i_2)\textit{-component} & \textit{where } m_1 = i_2 \neq m_2 = i_1 \\ \textit{of the reflection equation} & \textit{and } i_1 < i_2 \end{array} \right\}.$$

Lemma 7 (equation (X))**.** *Equation (X) follows from equation (X)$_{1/2}$.*

Similarly, when we interchange (u_1, i_1) with (u_2, i_2) in the equations of equation (VII), we obtain those of equation (VI).

Lemma 8 (equation (VII))**.** *Equation (VII) is equivalent to equation (VI).*

We investigate all cases which occur according to the order of the indices of the equations in equations (II), (III) and (V), then we have

Lemma 9 (equation (II))**.**

(i) If $m_1 < i < m_2$, then

$$\mathbf{e}\left[\frac{2}{N} m_1 u\right] c_i^{m_1}(u) \sim \mathbf{e}\left[\frac{2}{N} m_2 u\right] c_i^{m_2}(u).$$

(ii) If $m_1 < m_2 < i$ or $i < m_1 < m_2$, then

$$\mathbf{e}\left[\frac{2}{N}(m_1 + N) u\right] c_i^{m_1}(u) \sim \mathbf{e}\left[\frac{2}{N} m_2 u\right] c_i^{m_2}(u).$$

Lemma 10 (equation (III))**.**

(i) If $i_1 < i_2 < m$ or $m < i_1 < i_2$, then

$$\mathbf{e}\left[\frac{2}{N} i_1 u\right] c_{i_1}^{m}(u) \sim \mathbf{e}\left[\frac{2}{N} i_2 u\right] c_{i_2}^{m}(u).$$

(ii) If $i_1 < m < i_2$, then

$$\mathbf{e}\left[\frac{2}{N}(i_1 + N) u\right] c_{i_1}^{m}(u) \sim \mathbf{e}\left[\frac{2}{N} i_2 u\right] c_{i_2}^{m}(u).$$

Lemma 11 (equation (V))**.**

(i) If $m < i_1 < i_2$, $i_1 < m < i_2$, $i_2 < m < i_1$ or $i_2 < i_1 < m$, then

$$\mathbf{e}\left[\frac{2}{N} i_1 u\right] c_{i_1}^{i_2}(u) \sim \mathbf{e}\left[\frac{2}{N} m u\right] c_{i_2}^{m}(u).$$

(ii) If $m < i_2 < i_1$, then

$$\mathbf{e}\left[\frac{2}{N} i_1 u\right] c_{i_1}^{i_2}(u) \sim \mathbf{e}\left[\frac{2}{N}(m + N) u\right] c_{i_2}^{m}(u).$$

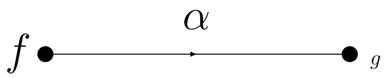

Figure 1. Graphical representation of $z^{\alpha} f(u) \sim g(u)$. We put functions $f(u)$ and $g(u)$ on the vertices, and the arrow from $f(u)$ to $g(u)$ with the variable α on it implies $z^{\alpha} f(u) \sim g(u)$.

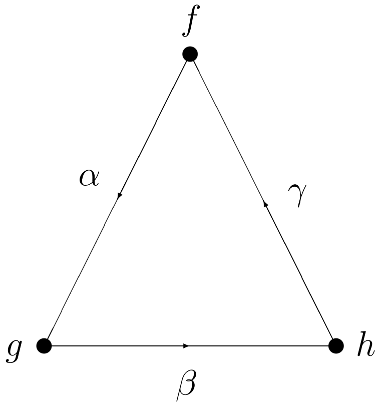

Figure 2. Compatibility among $z^{\alpha} f(u) \sim g(u)$, $z^{\beta} g(u) \sim h(u)$ and $z^{\gamma} h(u) \sim f(u)$. If there is a closed circuit along the directions of arrows and the sum of all variables on arrows concerned is zero, these relations are compatible to ensure that there exist all non-zero functions on all vertices. In the figure above, if $\alpha + \beta + \gamma = 0$, then there is a possibility that f, g and h are all non-zero at the same time, but if $\alpha + \beta + \gamma \neq 0$, then at least one of f, g and h should be zero. In the latter case, only two of f, g and h are capable of being non-zero simultaneously.

(iii) If $i_1 < i_2 < m$, then

$$\mathbf{e}\left[\frac{2}{N}(i_1 + N)u\right] c_{i_1}^{i_2}(u) \sim \mathbf{e}\left[\frac{2}{N}mu\right] c_{i_2}^{m}(u).$$

When we fix three integers $a, b, c \in \{0, 1, 2, \ldots, N-1\}$ such that $a < b < c$, lemmas 9–11 give the following twelve relations among six functions $c_b^a(u)$, $c_c^b(u)$, $c_a^c(u)$, $c_a^b(u)$, $c_b^c(u)$ and $c_c^a(u)$.

$$\begin{aligned}
&\mathbf{e}\left[\frac{2}{N}(b+N)u\right] c_a^b(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_a^c(u) &\qquad& \mathbf{e}\left[\frac{2}{N}au\right] c_b^a(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_b^c(u)\\
&\mathbf{e}\left[\frac{2}{N}(a+N)u\right] c_c^a(u) \sim \mathbf{e}\left[\frac{2}{N}bu\right] c_c^b(u) && \mathbf{e}\left[\frac{2}{N}bu\right] c_b^a(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_c^a(u)\\
&\mathbf{e}\left[\frac{2}{N}(a+N)u\right] c_a^b(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_c^b(u) && \mathbf{e}\left[\frac{2}{N}au\right] c_a^c(u) \sim \mathbf{e}\left[\frac{2}{N}bu\right] c_b^c(u)\\
&\mathbf{e}\left[\frac{2}{N}bu\right] c_b^a(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_a^c(u) && \mathbf{e}\left[\frac{2}{N}cu\right] c_c^a(u) \sim \mathbf{e}\left[\frac{2}{N}bu\right] c_a^b(u)\\
&\mathbf{e}\left[\frac{2}{N}(a+N)u\right] c_a^b(u) \sim \mathbf{e}\left[\frac{2}{N}cu\right] c_b^c(u) && \mathbf{e}\left[\frac{2}{N}cu\right] c_c^b(u) \sim \mathbf{e}\left[\frac{2}{N}(a+N)u\right] c_b^a(u)\\
&\mathbf{e}\left[\frac{2}{N}au\right] c_a^c(u) \sim \mathbf{e}\left[\frac{2}{N}bu\right] c_c^b(u) && \mathbf{e}\left[\frac{2}{N}bu\right] c_b^c(u) \sim \mathbf{e}\left[\frac{2}{N}au\right] c_c^a(u).
\end{aligned}$$

We examine the compatibility condition among these twelve equations and lemma 6, we obtain the following lemma.

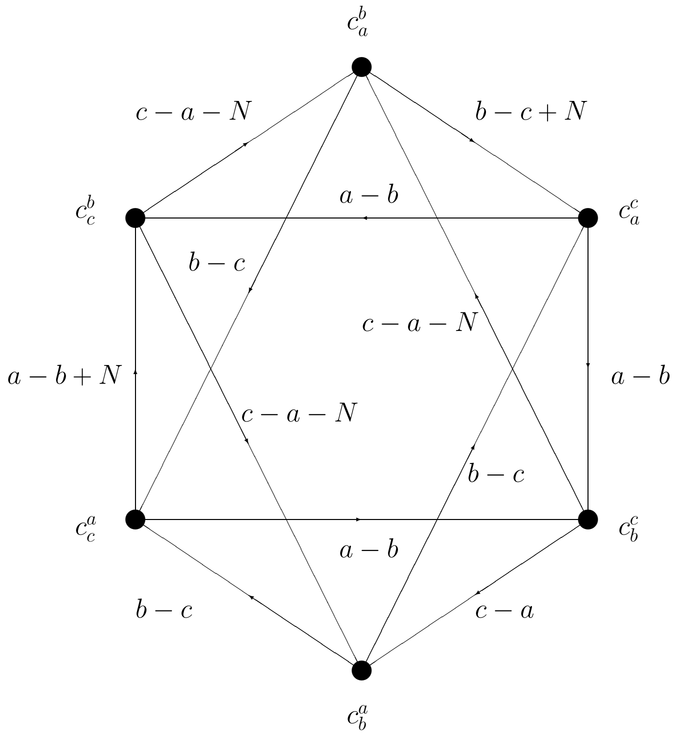

Figure 3. Graphical representation of the twelve relations among $\{c_b^a, c_c^b, c_a^c, c_a^b, c_b^c, c_c^a\}$.

In figure 3, there are six compatible combinations among $\{c_b^a, c_c^a, c_c^b, c_a^b, c_a^c, c_b^c\}$. Lemma 6 excludes three of them.

Lemma 12. *Let $a, b, c \in \{0, 1, 2, \dots, N-1\}$ be $a < b < c$, When we fix three integers a, b, $c \in \{0, 1, 2, \dots, N-1\}$, $a < b < c$, then there are six elements of $K(u) = \left(c^i_j(u)\right)_{ij}$ whose indices are a, b or c.*

$$C_{abc} := \{c_b^a, c_c^a, c_c^b, c_a^b, c_a^c, c_b^c\}.$$

There are three possibilities for maximal subsets in C_{abc} consisting of only non-zero elements,

$$\begin{aligned}
&(1) \quad \{c_b^a(u), c_c^a(u), c_c^b(u), c_a^b(u)\}\\
&(2) \quad \{c_b^a(u), c_c^a(u), c_a^c(u), c_b^c(u)\}\\
&(3) \quad \{c_c^b(u), c_a^b(u), c_a^c(u), c_b^c(u)\}.
\end{aligned}$$

When we write $z := \mathbf{e}\left[\frac{2}{N}u\right]$, they satisfy

$$\begin{aligned}
&(1) \quad \{c_b^a(u), c_c^a(u), c_c^b(u), c_a^b(u)\} \sim \{z^{-a-b}a(u), z^{-a-c}a(u), z^{-b-c+N}a(u), z^{-a-b}a(u)\}\\
&(2) \quad \{c_b^a(u), c_c^a(u), c_a^c(u), c_b^c(u)\} \sim \{z^{-a-b}a(u), z^{-a-c}a(u), z^{-a-c}a(u), z^{-b-c}a(u)\}\\
&(3) \quad \{c_c^b(u), c_a^b(u), c_a^c(u), c_b^c(u)\} \sim \{z^{-b-c}a(u), z^{-a-b-N}a(u), z^{-a-c}a(u), z^{-b-c}a(u)\}
\end{aligned}$$

for some function $a(u)$, where

$$\{f_1(u), f_2(u), f_3(u), f_4(u)\} \sim \{g_1(u), g_2(u), g_3(u), g_4(u)\}$$

means that $f_j(u)g_j(v) = f_j(v)g_j(u)$ for $j = 1, 2, 3, 4$.

Lemma 13. *Equation (IV) follows from equations (V) and (VI).*

Proof of lemma 13. An equation in equation (VI) is

$$\begin{aligned}
&\mathbf{s}\,[u_{12}]\,\mathbf{e}\,[v_{12}]\,A(i_1, m, i_2)\\
&\quad= -\mathbf{s}\,[v_{12}]\,\mathbf{e}\,[u_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(m-i_1)^* u_{12}\right] c^{i_1 m}_{i_1 i_2} - \mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^* u_{12}\right] c^{m i_1}_{i_2 i_1}\right)\\
&\qquad + 2\sqrt{-1}\,P(i_1, i_2, m)\mathbf{s}\,[u_{12}]\,\mathbf{s}\,[v_{12}]\,c^{m i_1}_{i_1 i_2}.
\end{aligned} \tag{3.10}$$

When we interchange u_1 with u_2 in (3.10), we have

$$\begin{aligned}
&\mathbf{s}\,[u_{12}]\,\mathbf{e}\,[v_{12}]\,B(i_1, i_2, m)\\
&\quad= \mathbf{s}\,[v_{12}]\,\mathbf{e}\,[u_{12}]\left(\mathbf{e}\left[-\frac{2}{N}(m-i_1)^* u_{12}\right] c^{m i_1}_{i_2 i_1} - \mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^* u_{12}\right] c^{i_1 m}_{i_1 i_2}\right)\\
&\qquad + 2\sqrt{-1}\,P(i_1, i_2, m)\mathbf{s}\,[u_{12}]\,\mathbf{s}\,[v_{12}]\,c^{i_1 m}_{i_2 i_1}
\end{aligned} \tag{3.11}$$

We substitute $A(i_1, m, i_2)$ in (3.10) and $B(i_1, i_2, m)$ in (3.11) into equation (IV), then we obtain

$$\mathbf{e}\left[-\frac{2}{N}(i_1-m)^*(u_1-u_2)\right] c^{m i_1}_{i_1 i_2} = \mathbf{e}\left[-\frac{2}{N}(i_1-i_2)^*(u_1-u_2)\right] c^{i_1 m}_{i_2 i_1}$$

which is the equation in equation (V). □

Summarizing the lemmas so far we obtained,

$$\begin{array}{cccccccccccc}
 & \mathrm{II} & & \mathrm{III} & & \mathrm{V} & \rightarrow & \mathrm{IV} & & & & \text{VIII : trivial}\\
 & & \nwarrow & \uparrow & \nearrow & & & \uparrow & & & & \\
\mathbf{I} & & & \textbf{Lemma 12} & & & & \mathbf{VI} & \mathbf{X_{1/2}} & \mathbf{XIII} & \mathbf{XIV} & \\
 & & \swarrow & & \searrow & & & \downarrow & & \downarrow & \downarrow & \\
 & \mathrm{IX} & & & & \mathrm{XV} & & \mathrm{VII} & & \mathrm{XII} & \mathrm{XI} &
\end{array}$$

we have

Theorem 2. *Solutions to the reflection equation for the critical $\mathbf{Z}_N$-symmetric vertex model of Belavin are determined by lemma 12, equations (I), (VI), $(X)_{1/2}$, (XIII) and (XIV).*

Because lemma 12, equations (I), (VI), (X), (XIII) and (XIV) are not concerned with the parameter η contained in $R(u, \eta)$ at all, we have proved that

Theorem 3. *Solutions to the reflection equation for the critical $\mathbf{Z}_N$-symmetric vertex model are independent of the parameter η of $R(u, \eta)$.*

4. Discussion

When we applied theorem 3 to the $N = 2$, the case of the eight-vertex model, the $2^4 = 16$ elements of the reflection equation reduce to five equations. Lemma 12, equations (I) and (VI) are empty when $N = 2$, and lemma 6 implies that $c^0_1(u) \sim c^1_0(u)$, namely

$$K(u) = \begin{pmatrix} c^0_0(u) & c^0_1(u) \\ c^0_1(u) & c^1_1(u) \end{pmatrix} = \begin{pmatrix} c^0_0(u) & \mu c(u) \\ \nu c(u) & c^1_1(u) \end{pmatrix}. \tag{4.12}$$

The equations we have to solve are

$$
\begin{array}{ll}
\text{Equation (X) (0, 1) :} & \mathbf{s}[u_1+u_2]\left(c_{10}^{10}-c_{01}^{01}\right)+\mathbf{s}[u_1-u_2]\left(c_{00}^{00}-c_{11}^{11}\right)=0 \\
\text{Equation (XIII) (0, 1) :} & \mathbf{s}[2u_2]\,c_{10}^{00}-\mathbf{s}[u_1+u_2]\,c_{01}^{00}-\mathbf{s}[u_1-u_2]\,c_{11}^{10}=0 \\
\text{Equation (XIII) (1, 0) :} & \mathbf{s}[2u_2]\,c_{01}^{11}-\mathbf{s}[u_1+u_2]\,c_{10}^{11}-\mathbf{s}[u_1-u_2]\,c_{00}^{01}=0 \\
\text{Equation (XIV) (0, 1) :} & \mathbf{s}[2u_2]\,c_{11}^{01}-\mathbf{s}[u_1+u_2]\,c_{11}^{10}-\mathbf{s}[u_1-u_2]\,c_{01}^{00}=0 \\
\text{Equation (XIV) (1, 0) :} & \mathbf{s}[2u_2]\,c_{00}^{10}-\mathbf{s}[u_1+u_2]\,c_{00}^{01}-\mathbf{s}[u_1-u_2]\,c_{10}^{11}=0.
\end{array}
$$

But the form of K-matrix in (4.12) implies that equations (XIV) (0, 1) and (XIV) (1, 0) follow from equations (XIII) (1, 0) and (XIII) (0, 1), respectively: we actually have only three equations to solve, and obtain the full $N=2$ solution,

$$
K(u)=f(u)\begin{pmatrix} k_0\mathbf{e}[u]-k_1\mathbf{e}[-u] & k_2\mathbf{s}[2u] \\ k_3\mathbf{s}[2u] & k_0\mathbf{e}[-u]-k_1\mathbf{e}[u] \end{pmatrix}
$$

where $f(u)$ is an arbitrary function. This solution has three free parameters besides the over-all factor and the spectral parameter. This result coincides with ones in [5] and [4].

The complete solution $K(u)$ for $N=3$ case is obtained on the basis of theorem 3, and the Segre three-fold appears as its parameter space. We report on this result in [13]. For cases $N>3$, the structures of solutions to the trigonometric reflection equation do not seem to be so simple. It is important to consider the elliptic case directly in these cases.

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